Strategic Complementarities in Matching Models

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Abstract

This paper describes an equilibrium search model with two types of workers producing two different intermediate goods. Although labour markets are perfectly segmented, strategic between sectors complementarity and strategic within sectors substitutability arise. This deeply changes the effects of labour market policies.

A welfare analysis is also conducted. Under constant returns to scale in the production technology, the so-called Hosios condition is sufficient to guarantee the efficiency of the decentralized equilibrium.

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1 Introduction

2 The model

2.1 Production Technology

Assume an economy with one final good (the numeraire), two intermediate goods sectors and two types -m and n- of infinitely-lived and risk-neutral workers. We can think for instance of a manifacturing sector and a sector of services. To keep the model as simple as possible, the goods markets are perfectly competitive. Each producer of an intermediate good hires only one type of worker. Moreover, every m-skilled employee produces one unit of the intermediate good m and every n-skilled employee produces one unit of the intermediate good n. Let Q_m (respectively, Q_n) denote the amount of the m intermediate good (respectively, the n intermediate good) used to produce the final good. The final good production function is homogeneous of degree one and written as:

$$Y = F(y_m Q_m, y_n Q_n), \quad \text{with } \frac{\partial F}{\partial Q_j} > 0 \text{ and } \frac{\partial^2 F}{\partial^2 Q_j} < 0, j \in \{m, n\}, \tag{1}$$

where y_m and y_n are technological parameters. We assume Inada conditions.[1] Furthermore, the two inputs are p-substitutes ($0 < \frac{\partial^2 F}{\partial Q_m \partial Q_n} < +\infty$). If E_m (respectively E_n) denotes the number of workers employed in the m (resp. n) sector, one obviously has $Q_m = E_m$ and $Q_n = E_n$. Let p_j denote the real price of the intermediate good j. Cost minimization leads to

$$p_j = f_j\left(E_j, E_i \mid y_j, y_i\right) \equiv \frac{\partial F(E_m y_m, E_n y_n)}{\partial E_j} \text{ with } j, i \in \{m, n\}, j \neq i \tag{2}$$

The price of each intermediate good depends negatively on the number of workers employed in that sector and positively on the number of workers employed in the other sector.

2.2 Search Technology

The model is markovian and developed in steady state. Time is continuous. Each type of worker can be either unemployed and receive an unemployment benefit b_j or works in his corresponding sector. At this stage, we do not allow workers' flows across the sectors. That is, every m-type worker can be hired only by firms in the m sector and

[1] $\lim_{Q_j \to 0} \frac{\partial F}{\partial Q_j} = +\infty$ and $\lim_{Q_j \to +\infty} \frac{\partial F}{\partial Q_j} = 0$.

the same holds for n-type workers. In this way, we have two different and perfectly segmented labor markets.

Due to various imperfections, the matching process is not instantaneous. A model of undirected search is therefore built where firms open skill-specific vacancies that are accessible both to participants and to the other job-seekers. The flow of hires, M_j is a function of an indicator of the number of job-seekers, U_j, and of the number of vacancies, V_j . The matching function is by assumption identical in both intermediate sectors and it is written respectively $M_j = m(U_j, V_j)$. The function $m(.,.)$ is assumed to be increasing, concave and homogeneous of degree 1.

At each moment, the timing of decisions is by assumption the following:

1. Firms post vacancies and this costs a fixed amount k_j per unit of time. Jobless workers search for a job.
2. At a certain endogenous rate, a firm meets a worker and the wage is bargained. The fall-back position of the workers is the intertemporal discounted utility of an unemployed entering state U_n, denoted by $V_{U,n}$
3. If an agreement is reached, production occurs in the intermediate-goods firms. These goods are sold to the final good firm and the total surplus is shared between the worker and the firm.
4. An exogenous fraction ϕ_n of the matches is destroyed. The workers who occupied these jobs enter unemployment and these jobs become vacant. As will soon be clear, workers have no incentive to quit.

Search intensity is exogenous and normalized to 1. Due to the constant return to scale in the matching process, the model can be developed in terms of tightness indicator, namely $\theta_j \equiv \frac{V_j}{U_j}$. The rate at which vacant jobs become filled is $q(\theta_j) \equiv M_j/V_j = m(\frac{1}{\theta_j}, 1), q'(\theta_j) < 0$. Every unemployed worker moves into employment according to a Poisson process with rate $\alpha(\theta_j) \equiv \frac{M_j}{U_j} = \theta_j\, q(\theta_j)$, with $\alpha'(\theta_j) > 0$. [2]

The elasticity of the probability q_j of filling a vacancy with respect to tightness θ_j is called $\eta(\theta_j) \equiv -\frac{d\,q(\theta_j)}{d\,\theta_j}\,\frac{\theta_j}{q(\theta_j)}$.

In steady state, the stocks of individuals in each position are constant. If we impose that L_j, the size of the labor force, is constant, equalities between entries and exits in every state determine the employment rate e_j in both labor markets:

$$e_j = \frac{\alpha(\theta_j)}{\phi_j + \alpha(\theta_j)},\ j \in \{m, n\}, \tag{3}$$

with $\frac{\partial e_j}{\partial \theta_j} > 0$.

[2] Moreover, $\lim_{\theta_j \to 0} q(\theta_j) = +\infty$ and $\lim_{\theta_j \to 0} \alpha(\theta_j) = +\infty$.

2.3 Preferences and Job Creation

Individuals are risk-neutral and have no access to capital market. Let r be the discount rate common to all agents. In steady state, the Bellmann equation representing the unemployment status is

$$rV_{U,j} = b_j + \alpha(\theta_j)(V_{E,j} - V_{U,j}). \tag{4}$$

This type of equation is standard in search literature. Being unemployed is similar to holding an asset that pays a dividend of b_j, the unemployment benefit, and it has a probability $\alpha(\theta_j)$ of being transformed in employment. In this case, the worker obtains $V_{E,j}$, the asset value of being employed, and he loses $V_{U,j}$

Similarly, the steady state discounted present value of employment can be written as:

$$rV_{E,j} = w_j + \phi_j(V_{U,j} - V_{E,j}), \tag{5}$$

where w_j is the wage bargained in the j-intermediate sector.

On the other side of the market, let $\Pi_{E,j}$ denote the firm's discounted expected return from an occupied job if the firm produces the jth intermediate good, namely it hires workers endowed with skill j. For simplicity, taxation is linear. Let $\tau_j\, w_j$ be the amount of taxes paid if the net wage is w_j ($\tau_j \geq 0$). The discounted expected return of vacant job is $\Pi_{V,j}$. We denote k_j the cost of posting a vacancy and of selecting applicants. For $j \in \{m, n\}$, the discounted expected returns satisfy the following conditions:

$$\begin{aligned} r\Pi_{E,j} &= p_j - (1+\tau_j)w_j + \phi_j\left(\Pi_{V,j} - \Pi_{E,j}\right), & (6)\\ r\Pi_{V,j} &= -k_j + q(\theta_j)\left(\Pi_{E,j} - \Pi_{V,j}\right). & (7) \end{aligned}$$

In equilibrium, firms open vacancies as long as they yield a positive expected return. Therefore, the equilibrium condition $\Pi_{V,j} = 0$, combined with (6) and (7), yields the following vacancy-supply curve for each j:

$$\frac{k_j}{q(\theta_j)} = \frac{p_j - (1+\tau_j)w_j}{r + \phi_j} \tag{8}$$

with $p_j = f_j\left(E_j,\, E_i \mid y_j,\, y_i\right), j \neq i$.

We can easily see that the vacancy-supply curve represents a decreasing relationship between the net wage and labor market tightness θ_j :

$$w_j = VS_j \equiv \frac{p_j - (r+\phi_j)\frac{k_j}{q(\theta_j)}}{1+\tau_j}, \tag{9}$$

with $\frac{\partial VS_j}{\partial \theta_j} < 0$ and $\frac{\partial VS_j}{\partial \theta_i} > 0$, $j \in \{m, n\}$, $j \neq i$.

Note that (9) depends on tightness indicators of both sectors, θ_j and θ_i, through the price of the intermediate good p_j. So, as θ_j increases, firms in sector j need to pay lower wages in order to equilibrate the zero profit condition for two reasons. First, because higher labor market tightness makes more difficult to fill a vacancy ($q(\theta_j)$ decreases). Second, because higher labor market tightness in sector j enhances employment through the steady state equation (3); more people employed in one intermediate sector lowers the marginal productivity of that intermediate good and j-firms' revenues are reduced (p_j goes down).

Moreover, an increase in labor market tightness in the other sector θ_i raises the marginal productivity of the intermediate good j, via equation (3) and the condition we imposed in (1) about the cross derivative.

2.4 Wage Formation

We assume that wages are bargained in both sectors of the economy and we impose Nash bargaining solution. This assumption is not essential, though rent sharing is crucial for the results. For each j, the Nash maximization program can be written as:

$$\max_{w_j} \left(V_{E,j} - V_{U,j}\right)^{\beta_j} \left(\Pi_{E,j} - \Pi_{V,j}\right)^{1-\beta_j}, \tag{10}$$

with $0 < \beta_j < 1$ representing the bargaining power of the workers. The first-order condition is equal to:

$$(1-\beta_j)(V_{E,j} - V_{U,j}) = \beta_j \, \frac{\Pi_{E,j} - \Pi_{V,j}}{1+\tau_j}, \tag{11}$$

Combining this last equation with (5) and (4) and taking into account of the free entry condition, $\Pi_{V,j} = 0$, we obtain:

$$w_j \;=\; rV_{U,j} \;+\; \beta_j \left(\frac{p_j}{1+\tau_j} - rV_{U,j} \right) \tag{12}$$

This equation has an intuitive interpretation. If the bargaining power of workers was equal to zero, they would get $w_j \;=\; rV_{U,j}$, i.e. their reservation wage. As β_j increases, a growing share of the ratio $p_j/(1+\tau_j)$ accrues to the employee.

Hereafter we follow the approach of Pissarides (2000). Free entry condition, $\Pi_{V,j} = 0$, combined with (7) and (11) yields:

$$(1-\beta_j)(V_{E,j} - V_{U,j}) = \beta_j \, \frac{k_j}{(1+\tau_j)\, q(\theta_j)} \tag{13}$$

Substituting this equality in (4), the expected discounted value of being unemployed can be written as:

$$rV_{U,j} \;=\; b_j + \,\alpha(\theta_j)\,\frac{\beta_j}{1-\beta_j}\,\frac{k_j}{(1+\tau_j)\,q(\theta_j)} \tag{14}$$

Finally, plugging (14) into (12) yields:

$$w_j \;= WS_j \;\equiv\; (1-\beta_j)b_j \,+\, \beta_j\,\theta_j\frac{k_j}{1+\tau_j} \,+\, \beta_j\frac{p_j}{1+\tau_j} \tag{15}$$

For each skill, the wage-setting curve $w_j = WS_j$ cannot be shown to be always upward sloping in $(\theta_j,\, w_j)$ space. The reason is that, as θ_j increases, the second term of the RHS increases too, but the productivity in the third term decreases.

Nevertheless, we can join the downward-sloping vacancy supply curve (9) and the wage-setting curve (15) and consider an implicit equation for θ_j, namely $\mathbb{G}_j \;= 0$:

$$\begin{aligned} &\mathbb{G}_j(\theta_j,\,\theta_i) \;\equiv\; VS_j \;-\; WS_j \;= \\ &\frac{1-\beta_j}{1+\tau_j}p_j \;-\; \frac{k_j}{1+\tau_j}\left(\frac{r+\phi_j}{q(\theta_j)}+\beta_j\theta_j\right) \;-\; (1-\beta_j)b_j \;=\; 0 \end{aligned} \tag{16}$$

with $j \in \{n, m\}, j \neq i$.

Note that $\mathbb{G}_j = 0$, the equilibrium condition in the labor market j, depends on θ_i only through the marginal productivity p_j. If we differentiate $\mathbb{G}_j$ with respect to θ_j, we obtain:

$$\frac{d\mathbb{G}_j}{d\theta_j} \;=\; \frac{1}{1+\tau_j}\left[(1-\beta_j)\frac{\partial p_j}{\partial E_j}\frac{\partial E_j}{\partial \theta_j} \;+ k_j(r+\phi_j)\frac{q'(\theta_j)}{q^2(\theta_j)} - \beta_j\right], \tag{17}$$

that is negative because all the terms inside the square brackets are negative[3].

Hence, for any θ_i, it exists a θ_j such that the equation $\mathbb{G}_j = 0$ is verified[4].

Finally, for reasons of simplicity in notation we define

$$A_j \equiv \frac{1}{1+\tau_j}\left[k_j(r+\phi_j)\frac{q'(\theta_j)}{q^2(\theta_j)} - \beta_j\right] \;\text{ and }\; B_j \equiv \frac{1}{1+\tau_j}\left[(1-\beta_j)\frac{\partial p_j}{\partial E_j}\frac{\partial E_j}{\partial \theta_j}\right].$$

with $j \in \{n, m\}, j \neq i$. Both A_j and B_j are negative; A_j is the part of the derivative in (17) that stems from the search technology ($q'(\theta_j)$ negative), whereas B_j is the part that stems from the production technology (p_j that negatively depends on θ_j).

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[3] Recall that $\partial p_j/\partial E_j$ is negative because we imposed the marginal productivity being decreasing in its input and that $\partial E_j/\partial \theta_j$ is positive from equation (3).

[4] To see this, note that $\lim_{\theta_j\to 0}\mathbb{G}_j(\theta_j,\theta_i) = +\infty$ and that $\lim_{\theta_j\to +\infty}\mathbb{G}_j(\theta_j,\theta_i) = -\infty$. So, since $\mathbb{G}_j$ is continuous and always decreasing in θ_j, it exists a θ_j such that $\mathbb{G}_j = 0\ \forall\theta_i$.

2.5 Existence and Uniqueness of the Equilibrium

The steady-state equilibrium values of tightness in both sectors are characterized by a system of two equations:

$$\begin{cases} \mathbb{G}_n(\theta_n, \theta_m) = 0 \\ \mathbb{G}_m(\theta_m, \theta_n) = 0 \end{cases} \tag{18}$$

Every function $\mathbb{G}_j$ represents the equilibrium condition in the j-th labor market. The novelty with respect to the standard matching model lies on the link between the two intermediate sectors: each labor market depends on tightness of the other one through the price (the marginal productivity) of the intermediate good.

Proposition 1. *The system (18) has a unique equilibrium.*

Proof. For the complete proof we refer to Appendix 1. Here we simply give a sketch of it, stressing the most important passages:

1. *Intercepts of* $\mathbb{G}_m(\theta_m, \theta_n) = 0$ *and* $\mathbb{G}_n(\theta_n, \theta_m) = 0$.

   Consider $\mathbb{G}_m = 0$ and $\mathbb{G}_n = 0$ as two explicit functions in the (θ_m, θ_n) space (θ_m is measured along the vertical axis and θ_n along the horizontal one).We show in Appendix 1 that θ_n in $\mathbb{G}_n(\theta_n, \theta_m = 0) = 0$ is positive, and θ_m in $\mathbb{G}_m(\theta_m, \theta_n = 0) = 0$ is also positive.

2. $\mathbb{G}_m = 0$ *and* $\mathbb{G}_n = 0$ *are monotonously increasing in* (θ_m, θ_n) *space.*

   We differentiate $\mathbb{G}_j$ with respect to θ_i :

   $$\frac{\partial \mathbb{G}_j}{\partial \theta_i} = \frac{1-\beta_j}{1-\tau_j} \frac{\partial p_j}{\partial E_i} \frac{\partial E_i}{\partial \theta_i}. \tag{19}$$

   This term is positive because the intermediate inputs are p-substitutes and from equation (3). Hence, using (17) and (19) and applying the implicit function theorem, we have:

   $$\left.\frac{d\,\theta_j}{d\,\theta_i}\right|_{\mathbb{G}_j=0} = -\frac{\partial \mathbb{G}_j / \partial \theta_i}{\partial \mathbb{G}_j / \partial \theta_j} > 0 \tag{20}$$

   with $j \in \{n, m\}, j \neq i$.

3. *Existence and Uniqueness.*

   See figure 2. From point 1 and 2, it's straightforward to conclude that, if

   $$\left.\frac{d\,\theta_m}{d\,\theta_n}\right|_{\mathbb{G}_n=0} > \left.\frac{d\,\theta_m}{d\,\theta_n}\right|_{\mathbb{G}_m=0} \forall(\theta_n, \theta_m) \tag{21}$$

then an equilibrium pair (θ_n, θ_m) exists and is unique . This indeed we prove in Appendix 1.

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3 Strategic complementarity and strategic substitutability

The model we have developed so far is not new in the search-matching literature (see for instance Acemoglu (2001) and the textbook of Cahuc and Zylberberg (2001) pages 648-657). The idea of constructing a set-up where two or more different sectors are present - typically a low-skilled and a high-skilled one - linked together through a final good production function, is widespread also in that macroeconomic literature trying to introduce search frictions in pure RBC models. However, to the best of our knowledge, this paper is the first one devoted to studying the analytical effects that are on work in such framework. These effects crucially depend on the search and production technologies we presented in section 1. Therefore, we divide our analysis in two parts.

3.1 Production technology

Consider the equilibrium system (18). When firms open new vacancies in one sector j, labor market tightness θ_j increases and the productivity p_i in the other intermediate sector i increases too. This effect occurs because higher θ_j enhances employment E_j through (3) and because the cross derivative in the final good production function is positive: $0 < \frac{\partial^2 F}{\partial Q_m \partial Q_n} < +\infty$. Since in equilibrium the price of the intermediate good is equal to its marginal productivity, firms in sector i will get higher revenues and, to restore the zero profit condition, new vacancies in this sector will be opened too.

Following the approach of Cooper (1999), we say that our economy exhibits *strategic complementarity* : the best response of one agent to an increase in the activity of the others is to increase his own activity. In other terms, consider (8), the zero profit condition in sector j. When the activity of firms in sector i increases (that is, when new vacancies are opened in sector i), the price of the intermediate good j, p_j, goes up and, in turn, the best response of firms in j is to increase their activity (more vacancies V_j are posted). In terms of tightness indicator, strategic complementarity is equivalent to inequality (20). Therefore, the existence of strategic complementarities can be seen in figure 2, where both curves are upward sloping. Such curves can be seen as the reaction functions of one sector of the economy with respect to the other one.

However, this is not the only type of interdependent behaviour present in our economy. When a new vacancy is opened in sector j, labor market tightness θ_j and, in turn, the level of employment E_j increase too. That has a negative effect on the price of the intermediate good p_j, that decreases[5]. Lower price of the intermediate good j will diminish the convenience of other firms to open new vacancies in sector j. In other words, the increase in the activity of firms in sector n induces other firms to create *less* vacancies in sector n.

Borrowing again the terminology from Cooper (1999), we say that the economy exhibits *strategic substitutability;* the term B_j in (17) captures such presence in our model. If p_j was exogenous, the derivative of $\mathbb{G}_j$ with respect to θ_j still would be negative (A_j is lower than zero); but it would be *less* negative. In our framework, every vacancy created in sector j decreases both the probability for other vacancies in j to be filled both the productivity of their jobs .

To sum up our results, the introduction of a final good production function, by making endogenous the productivity of each intermediate market, produces two opposite effects. Every more vacancy opened in one sector at the same time worsens the profitability of this sector (inducing firms to open less vacancies), and improves the profitability of the other one (inducing firms to post more vacancies there). Strategic *between sectors* complementarity and strategic *within sectors* substitutability coexist in our model.

3.2 Search technology

Although the primary task of this paper is to focus on the effects arising from the production technology, it is also useful to study how the search technology affects the decentralized equilibrium outcome. As Cooper (1999) observed, any search-matching model gives rise to strategic complementarity and substitutability[6]. Tobin (1972) many years ago wrote:

> A worker deciding to join a queue or stay in one considers the probabilities of getting a job, but not the effects of his decision on the probabilities that others face....

This quotation makes clear how workers and firms behave in matching models; strategic substitutability is present because a firm deciding to post a new vacancy lowers the probability that other vacancies can be filled and, therefore, it induces other firms to decrease their activity (that is, less vacancies are created). Note that such strategic

[5]Recall that we assumed $\frac{\partial^2 F}{\partial^2 Q_j} < 0, j \in \{m, n\}$.

[6]In his book, Cooper analyzes only the Diamond (1984) model of search and unemployment, but his reasoning can be extended to all the models with a matching technology.

substitutability must be distinguished from that we discussed in the previous subsection. There, strategic substitutability is generated by the production technology (p_j decreasing in E_j). Here, it stems from the search technology (the probability of a vacancy to be filled $q(\theta_j)$ negatively depends on V_j). In our model, the former is represented by term B_j, the latter by term A_j.

If in our matching framework the labor force participation rate was endogenous, the symmetry between search and production technology would be perfect. Another strategic substitutability would arise, because an agent deciding to enter the labour market would negatively affect the maximizing behaviour of other agents outside the labour force. Moreover, also strategic complementarity between the sides of the market would arise: a new vacancy posted would induce agents to enter the labour market and the other way round.

3.3 Comparative Statics and the Multiplier

To better understand the policy implications of strategic complementarity and substitutability, we start analyzing what happens when a policy parameter changes. Suppose for instance a reduction in the unemployment benefits in sector n, b_n.[7] We consider first the standard set-up where productivity is fixed; then we study our model where it is endogenous; finally, we compare the results.

A reduction in b_n affects equation $\mathbb{G}_n = 0$: lower unemployment benefits reduce workers' outside option in the Nash bargain and the wage w_n decreases. Hence, firms will be induced to post more vacancies in the n sector and θ_n will go up, equilibrating the labor market. This is the standard mechanism at work in simple matching models: when p_n is exogenous, it does not exist any link between the two intermediate sectors n and m and both the strategic complementarity and the strategic substitutability generated by the production function are equal to zero. Only strategic substitutability that come from the matching technology is present. Therefore, in order to see analytically the effect of a marginal variation in b_n, we need to consider only equation (16) in the case of $j = n$. Differentiating $\mathbb{G}_n = 0$ with respect to θ_n and b_n and applying the implicit function theorem, we obtain:

$$\left.\frac{d\,\theta_n}{d\,b_n}\right|_{\mathbb{G}_n^{\bar{p}}=0} = -\frac{\partial \mathbb{G}_n^{\bar{p}}/\partial b_n}{\partial \mathbb{G}_n^{\bar{p}}/\partial \theta_n} = \frac{1-\beta_n}{A_n} < 0, \tag{22}$$

where the index $\bar{p}$ indicates that we are dealing with the case of exogenous productivity[8]. As expected, such derivative is negative: a decrease in the level of unemployment

[7] For simplicity we do not introduce a budget balanced condition of the State.

[8] Note that:

$$\frac{d\mathbb{G}_j}{d\theta_j} = A_j + B_j < \frac{d\mathbb{G}_j^{\bar{p}}}{d\theta_j} = A_j < 0,$$

benefits in n increases labor market tightness θ_n and so E_n.

Consider now the hypothesis of endogenous productivity; in this case, to perform comparative statics we have to consider the system (18). We differentiate such system with respect to θ_n, θ_m and b_n and again apply the implicit function theorem. Using Cramer's rule, we obtain:

$$\left.\frac{d\theta_n}{db_n}\right|_{system(18)} = -\frac{det\begin{bmatrix} \partial\mathbb{G}_n/\partial b_n & \partial\mathbb{G}_n/\partial\theta_m \\ 0 & \partial\mathbb{G}_m/\partial\theta_m \end{bmatrix}}{det\begin{bmatrix} \partial\mathbb{G}_n/\partial\theta_n & \partial\mathbb{G}_n/\partial\theta_m \\ \partial\mathbb{G}_m/\partial\theta_n & \partial\mathbb{G}_m/\partial\theta_m \end{bmatrix}}. \tag{23}$$

Dividing the denominator and the numerator by $\frac{\partial\mathbb{G}_n}{\partial\theta_n}\frac{\partial\mathbb{G}_m}{\partial\theta_m}$ and using (20), we get the following expression:

$$\left.\frac{d\theta_n}{db_n}\right|_{system(18)} = -\frac{\partial\mathbb{G}_n/\partial b_n}{\partial\mathbb{G}_n/\partial\theta_n}\cdot\frac{1}{1-\left.\frac{d\theta_m}{d\theta_n}\right|_{\mathbb{G}_m=0}\cdot\left.\frac{d\theta_n}{d\theta_m}\right|_{\mathbb{G}_n=0}} < 0 \tag{23b}$$

Expressed in this way, we can interpret more easily the results of the comparative statics. The first term in (23b) is negative. It represents the effects that arise *within* the intermediate sector n when b_n marginally changes. Note that this first term is *greater* than (22), because $-\partial\mathbb{G}_n/\partial b_n = -\partial\mathbb{G}_n^{\bar{p}}/\partial b_n = 1-\beta_n$, and at the denominator $\partial\mathbb{G}_n/\partial\theta_n$ is lower than $\partial\mathbb{G}_n^{\bar{p}}/\partial\theta_n$. The reason lies on the fact that in $\partial\mathbb{G}_n/\partial\theta_n$ there is also the component B_j. Therefore, the first term in (23b) is greater than (22) because of the strategic (production) substitutability within sector n. This effect tends to increase (reduce) the negative (positive) impact on θ_n caused by an increase (a decrease) in the unemployment benefits.

On the other hand, the second term in (23b) captures the effects that intervene *between* the two sectors when b_n marginally changes. Both derivatives at the denominator are positive (see equation (20)): they represent the strategic complementarities existing between sector n and sector m. The denominator is also positive and lower than one[9]. Therefore, the second term in (23b) is greater than one. It is a *multiplier,* because it amplifies the effects of a shock on a single sector (in this case a change in b_n) through the induced response of the other sector: higher unemployment benefits in n reduce the number of vacancies V_n and so θ_n; this lowers productivity p_m and, consequently, V_m and θ_m. In turn, lower θ_m will reduce θ_n even more. It's worthwhile to notice also that, for the existence of a multiplier, it's necessary that strategic

with $j \in \{n, m\}$.

[9]To see this, note that showing that the denominator in the second term of (23b) is greater than zero is equivalent to proving inequality (21). Such inequality is proved in Appendix 1.

complementarities are present in both sectors. This makes sense: the decrease in θ_m caused by higher unemployment benefits b_n must in turn affect θ_n in order to have a multiplicative impact in n sector.

We can see graphically the effects of a decrease in b_n looking at figure 1: the curve $\mathbb{G}_n = 0$ shifts to the right and in the new equilibrium point E' both θ_n and θ_m are higher[10].

Moreover, we can say that two opposite effects intervene in our model: the presence of strategic substitutability in the first term tends to mitigate the impact of a labor market policy on tightness and employment, whereas strategic complementarity in the second term tends to enhance it. To see which of two effects prevails, we compute the derivatives in (23b) and, after some easy algebra, we obtain:

$$\left.\frac{d\theta_n}{db_n}\right|_{system(18)} = \frac{1-\beta_n}{A_n} \cdot \frac{A_m + \frac{1-\beta_m}{1+\tau_m}\frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m}}{A_m + \frac{1-\beta_n}{1+\tau_n}\frac{\partial p_n}{\partial E_n}\frac{\partial E_n}{\partial \theta_n}\frac{A_m}{A_n} + \frac{1-\beta_m}{1+\tau_m}\frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m}}. \tag{23c}$$

Since all the terms in the second ratio are negative, we conclude:

$$\left.\frac{d\,\theta_n}{d\,b_n}\right|_{\mathbb{G}_n^{\bar{p}}=0} < \left.\frac{d\theta_n}{db_n}\right|_{system(18)}$$

The magnitude of strategic substitutability within sector n overcompensates strategic complementarity between the two sectors. To conclude:

Proposition 2.
The introduction of endogenous productivity through a final good production function in a simple two-sectors Pissarides model has a twofold effect:

1. *Strategic substitutability: it diminishes in absolute value the effects of any labour market policy - targeted for a single sector - on tightness indicator and employment in that sector.*

2. *Strategic complementarity: it affects the level of tightness indicator and employment in the other sector; such effect has the same sign of point 1.*

Proof. For point 1, look at (22) and (23c). Point 2 can be seen analytically by using

[10] In the hypothesis of zero strategic complementarities, the equation $\mathbb{G}_n = 0$ would be a vertical line and $\mathbb{G}_m = 0$ an horizontal line in the (θ_m, θ_n) space. A decrease in b_n would shift $\mathbb{G}_n = 0$ to the right and the the new equilibrium point would present higher θ_n but the same value of θ_m.

again the implicit function theorem for system (18):

$$\frac{d\theta_m}{db_n}\Big|_{system(18)} = -\frac{det\begin{bmatrix} \partial\mathbb{G}_n/\partial\theta_n & \partial\mathbb{G}_n/\partial b_n \\ \partial\mathbb{G}_m/\partial\theta_n & 0 \end{bmatrix}}{det\begin{bmatrix} \partial\mathbb{G}_n/\partial\theta_n & \partial\mathbb{G}_n/\partial\theta_m \\ \partial\mathbb{G}_m/\partial\theta_n & \partial\mathbb{G}_m/\partial\theta_m \end{bmatrix}} < 0. \tag{24}$$

Computing the determinants, we can easily see that such derivative is negative. Otherwise, we can conclude about the negative sign of (24) without going through the algebraic passages; since policy in sector n affects employment in m only through productivity p_m, and p_m depends positively on E_n, any change in sector n that raises (lowers) E_n will also raise (lower) p_m and so E_m. Looking at figure 1 we see that, when the curve $\mathbb{G}_n = 0$ shifts to the right, both θ_n and θ_m increase. ∎

4 Welfare analysis

4.1 Efficiency

Since the work of Hosios (1990), it has been well known that a simple matching model with Nash bargaining does not necessarily reach efficiency [11]. More precisely, the so-called Hosios condition states that if the bargaining power of workers β equals the absolute value of $\eta(\theta)$, the elasticity of the probability q of filling a vacancy with respect to tightness, the search externalities are internalized by the ex post Nash bargain. When these variables are different, the decentralized equilibrium does not ensure an efficient allocation of the two inputs (unemployment and vacancies) in the matching technology. For instance, with $\beta > \eta(\theta)$ the equilibrium unemployment is inefficiently high.

The source of such inefficiency lies on the way in which the wage formation is formalized. In the decentralized economy, the extent of substitution between unemployment and vacancies is not governed by the matching technology but by the bargaining solution. Imposing $\beta = \eta(\theta)$ is therefore equivalent to impose that the substitution between unemployment and vacancies is adjusted according to the matching function.

In this section, we address the efficiency concern in our model, where strategic productivity complementarity and substitutability are added and therefore new externalities arise. As usual in the literature, we consider the simplest set-up: the economy is in steady state, the discount rate r tends to 0, and taxation is ignored ($\tau_m = \tau_n = 0$).

In Appendix 2 we show that a social planner that *would ignore distributional issues* would maximize:

[11] For efficiency we mean the maximization of the steady state value of the social output.

$$
\begin{aligned}
&\max_{\theta_n,\,\theta_m} \{p_n E_n \,+ b_n(L_n - E_n) \,- k_n V_n \,+\, p_m E_m \,+ b_m(L_m - E_m) \,- k_m V_m\} \\
&s.t.\;\; E_j = \frac{\alpha(\theta_j)}{\phi_j + \alpha(\theta_j)}\, L_j\,,\;\; j \in \{m,n\}.
\end{aligned} \tag{25}
$$

Maximize the social output for the social planner is equivalent to maximize the aggregate revenues of the intermediate firms and the value of leisure for the unemployed workers, net to the cost of opening vacancies, in both sectors.

The F.O.Cs of this problem are the following:

$$
\begin{aligned}
&L_m \, \frac{\phi_m \alpha'(\theta_m)[p_m - b_m + k_m\theta_m] \,-\, k_m\phi_m[\alpha(\theta_m) + \phi_m]}{[\alpha(\theta_m) + \phi_m]^2} + \\
&+\; E_m \, \frac{\partial p_m}{\partial E_m}\, \frac{\partial E_m}{\partial \theta_m} \,+\; E_n \, \frac{\partial p_n}{\partial E_m}\, \frac{\partial E_m}{\partial \theta_m} \;=\; 0
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&L_n \, \frac{\phi_n \alpha'(\theta_n)[p_n - b_n + k_n\theta_n] \,-\, k_n\phi_n[\alpha(\theta_n) + \phi_n]}{[\alpha(\theta_n) + \phi_n]^2} + \\
&+\; E_n \, \frac{\partial p_n}{\partial E_n}\, \frac{\partial E_n}{\partial \theta_n} \,+\; E_m \, \frac{\partial p_m}{\partial E_n}\, \frac{\partial E_n}{\partial \theta_n} \;=\; 0
\end{aligned} \tag{27}
$$

The first term in (26) represents the net marginal gain of increasing θ_m with p_m fixed. This term is the usual one obtained in the social planner optimization problem.

In the second line of (26) we have two terms. The first one represents the marginal productivity loss suffered by the m-sector when θ_m increases marginally. It's a negative externality because a firm deciding to open a new vacancy in the m sector does not take into account of the decrease in productivity it induces. The second term is the marginal productivity gain that firms in the n sector obtain if θ_m increases. Again this is a (positive) externality, since m-firms do not take into account that a new vacancy opened in their sector benefits firms in sector n through an increase in p_n. Equation (27) is symmetric; we simply have the same marginal gains/losses referred to the n-sector.

If we focus on the second line of (26), we observe that:

$$
E_m \, \frac{\partial p_m}{\partial E_m}\, \frac{\partial E_m}{\partial \theta_m} \,+\, E_n \, \frac{\partial p_n}{\partial E_m}\, \frac{\partial E_m}{\partial \theta_m} \;=\; \frac{\partial E_m}{\partial \theta_m} \left(E_m \, \frac{\partial p_m}{\partial E_m} \,+\, E_n \, \frac{\partial p_n}{\partial E_m} \right) \;=\; 0 \tag{28}
$$

This is true for every constant return to scale production function[12].

[12]We are exploiting the following property of a CRS production function $Y = F(y_m\, Q_m, y_n\, Q_n)$: $Y = \frac{\partial F}{\partial Q_n} Q_n + \frac{\partial F}{\partial Q_m} Q_m$ and, differentiating with respect to Q_m, $\frac{\partial^2 F}{\partial^2 Q_m} Q_m + \frac{\partial^2 F}{\partial Q_m \partial Q_n} Q_n \;=\; 0$.

Of course, the same holds for the second line in equation (27). The economic intuition of this result is the following. As we have seen in section 3, when a firm decides to open a new vacancy in the intermediate sector j , strategic productivity complementarity and strategic productivity substitutability arise at the same time. Both effects are externalities, because firms do no take into account of them when they decide to post a vacancy or not. What equation (28) tells us is that these two externalities are equal in absolute value. The marginal loss in the aggregate revenues in sector j ($E_j \frac{\partial p_j}{\partial E_j} \frac{\partial E_j}{\partial \theta_j}$) is totally offset by the marginal gain in the aggregate revenues in sector i ($E_i \frac{\partial p_i}{\partial E_j} \frac{\partial E_j}{\partial \theta_j}$).

Proposition 3. *Consider the system (18) with $r \to 0$ and $\tau_m = \tau_n = 0$. The Hosios conditions, $\beta_j = \eta(\theta_j)$ (with $j \in \{n, m\}$), are sufficient for the decentralized equilibrium to be efficient.*

Proof. Using (28), the F.O.C. (26) becomes:

$$\phi_m \alpha'(\theta_m)[p_m - b_m + k_m \theta_m] \; - \; k_m \phi_m [\alpha(\theta_m) + \phi_m] \; = \; 0.$$

Rearranging the terms and knowing that $\alpha'(\theta_m) \; = \; q(\theta_m) \; [1 - \eta(\theta_m)]$, we get:

$$q(\theta_m) \, [1 - \eta(\theta_m)] \, (p_m - b_m) \; = \; k_m \, [\alpha(\theta_m)\eta(\theta_m) + \phi_m] \, .$$

With $\beta_m \; = \; \eta(\theta_m)$, this equation is equivalent to (16). (Provided that $r \to 0$ and $\tau_m = 0$.) Obviously, the same passages lead to $\beta_n = \eta(\theta_n)$ in sector n.

■

4.2 Distributional issues

Proposition 2 states that, since the productivity externalities cancel out each other, to reach efficiency only the search externalities have to be internalized. The Hosios condition applied both in sector n and in sector m ensures it.

In this light, the remark we did some lines above about a social planner that *does not care about distributional issues* becomes significant. In fact, the productivity externalities that are present in this model do not affect the efficient value of labor market tightness that a social planner would choose: if p_l and p_h were exogenous (i.e. if the only link between the two intermediate sector disappeared), then a social planner without distributive concern would select exactly the same value of θ_m and θ_n.

What the introduction of productivity externalities change, it's the distribution of the resources between the sectors. Put in other terms, the four productivity externalities present in (26) and (27) can be viewed as a mechanism that shifts resources from one sector to the other. The net marginal amount of resources that accrues to sector m by means of productivity externalities is:

$$E_m \left(\frac{\partial p_m}{\partial E_m} \frac{\partial E_m}{\partial \theta_m} + \frac{\partial p_m}{\partial E_n} \frac{\partial E_n}{\partial \theta_n} \right), \tag{26b}$$

while the net marginal amount of resources going to sector n is equal to:

$$E_n \left(\frac{\partial p_n}{\partial E_m} \frac{\partial E_m}{\partial \theta_m} + \frac{\partial p_n}{\partial E_n} \frac{\partial E_n}{\partial \theta_n} \right). \tag{27b}$$

As we showed above, the sum of (26b) and (27b) is equal to zero; so expressions in (26b) and (27b) either have the same absolute value with opposite sign or are both equal to zero. If it is true the latter, we have no transfer of resources between sectors. Otherwise, with (26b) positive (negative), resources go from sector m to sector n (from sector n to sector m).

Imagine a social planner that prefers one sector to the other. Then, in this last case, it's no longer true that he would select the same values of θ_m and θ_n that he would have chosen in the case of exogenous productivity.

5 Conclusions

Appendix 1: Existence and uniqueness of the steady-state equilibrium

Consider $\mathbb{G}_m$ and $\mathbb{G}_n$. Think of them as two explicit functions in the (θ_m, θ_n) space (θ_m is measured along the vertical axis and θ_n along the horizontal one).
We start studying the intercept of $\mathbb{G}_j$ when $\theta_i \to 0$ with $j \in \{n, m\}, j \neq i$:

$$\lim_{\theta_i \to 0} \mathbb{G}_j(\theta_j,\, \theta_i) = \frac{1-\beta_j}{1+\tau_j} \lim_{\theta_i \to 0} p_j \;-\; \frac{k_j}{1+\tau_j}\left(\frac{r+\phi_j}{q(\theta_j)} + \beta_j \theta_j\right) \;-\; (1-\beta_j) b_j \;=\; 0$$

Looking at equation (3) we know that $\lim_{\theta_j \to 0} E_j = 0$. So, since we imposed that $0 < \frac{\partial^2 F}{\partial Q_m \partial Q_n} < +\infty$, we conclude that the limit above exists and it's positive.
Hence, we have that the intercept of $\mathbb{G}_n = 0$ along the horizontal axis is positive and the intercept of $\mathbb{G}_m = 0$ along the vertical axis is also positive.
Moreover, $\mathbb{G}_n = 0$ (resp. $\mathbb{G}_m = 0$)can be seen as implicitly defining a monotonously increasing relationship between θ_n and θ_m (resp., θ_m and θ_n).
Then, it's straightforward to conclude that, if:

$$\left.\frac{d\,\theta_m}{d\,\theta_n}\right|_{\mathbb{G}_n=0} > \left.\frac{d\,\theta_m}{d\,\theta_n}\right|_{\mathbb{G}_m=0} \forall(\theta_n, \theta_m) \tag{21}$$

then an equilibrium pair (θ_l, θ_h) exists and is unique
From (17) and (19), one derives:

$$\left.\frac{d\,\theta_m}{d\,\theta_n}\right|_{\mathbb{G}_n=0} = -\frac{\partial \mathbb{G}_n/\partial \theta_n}{\partial \mathbb{G}_n/\partial \theta_m} = -\frac{(1-\beta_n)\frac{\partial p_n}{\partial E_n}\frac{\partial E_n}{\partial \theta_n} \;+ k_n(r+\phi_n)\frac{q'(\theta_n)}{q^2(\theta_n)} - \beta_n}{(1-\beta_n)\frac{\partial p_n}{\partial E_m}\,\frac{\partial E_m}{\partial \theta_m}} \tag{A1.1}$$

$$\left.\frac{d\,\theta_m}{d\,\theta_n}\right|_{\mathbb{G}_m=0} = -\frac{\partial \mathbb{G}_m/\partial \theta_n}{\partial \mathbb{G}_m/\partial \theta_n} = -\frac{(1-\beta_m)\frac{\partial p_m}{\partial E_n}\,\frac{\partial E_n}{\partial \theta_n}}{(1-\beta_m)\frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} \;+ k_m(r+\phi_m)\frac{q'(\theta_m)}{q^2(\theta_m)} - \beta_m} \tag{A1.2}$$

We want to check (21). So we multiply the numerator of (A1.1) with the denominator of (A1.2) and the numerator of (A1.2) with the denominator of (A1.1): we get six positive terms in the LHS and only one positive term in the RHS. For (21) to hold, the six positive terms in the LHS must be larger than the term in the RHS.
One of the terms in the LHS is:

$$(1-\beta_n)\,(1-\beta_m)\,\frac{\partial p_n}{\partial E_n}\frac{\partial E_n}{\partial \theta_n}\,\frac{\partial p_m}{\partial E_m}\frac{\partial E_m}{\partial \theta_m} \tag{A1.3}$$

The positive term in the RHS is:

$$(1-\beta_n)(1-\beta_m)\frac{\partial p_n}{\partial E_m}\frac{\partial E_m}{\partial \theta_m}\frac{\partial p_m}{\partial E_n}\frac{\partial E_n}{\partial \theta_n} \tag{A1.4}$$

Indeed, expressions (A1.3) and (A1.4) are equal because of Eulero's formula for linear homogeneous function:

$$\frac{\partial^2 F}{\partial^2 Q_n}\frac{\partial^2 F}{\partial^2 Q_m} = \left[\frac{\partial^2 F}{\partial Q_m \partial Q_n}\right]^2$$

Then, inequality (21) is verified. We have two increasing functions in the (θ_m, θ_n) space. Moreover, the slope of the function with a positive intercept in the horizontal axis ($\mathbb{G}_n = 0$) is always higher than the slope of the function that has a positive intercept in the vertical axis ($\mathbb{G}_m = 0$). An equilibrium exists and it's unique.

Appendix 2: Derivation of the Social Output Function

TO BE ADDED (look at the handwritten note of Bruno)

Figure 1: Existence or Uniqueness